

Confirming and extending the hypothesis of universality in sandpiles

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Stochastic sandpiles self-organize to an absorbing-state critical point with scaling behavior different from directed percolation (DP) and characterized by the presence of an additional conservation law. This is usually called the C-DP or Manna universality class. There remains, however, an exception to this universality principle: a sandpile automaton introduced by Maslov and Zhang, which was claimed to be in the DP class despite the existence of a conservation law. We show, by means of careful numerical simulations as well as by constructing and analyzing a field theory, that (contrarily to what was previously thought) this sandpile is also in the C-DP or Manna class. This confirms the hypothesis of universality for stochastic sandpiles and gives rise to a fully coherent picture of self-organized criticality in systems with conservation. In passing, we obtain a number of results for the C-DP class and introduce a strategy to easily discriminate between DP and C-DP scaling.

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I. INTRODUCTION

Aimed at shedding some light on the origin of scale invariance in many contexts in nature, different mechanisms for *self-organized criticality* (SOC) were proposed during the last two decades following the seminal work by Bak and others [1–3]. Sandpiles, ricepiles, and earthquake toy models become paradigmatic examples capturing the essence of self-organization to scale-invariant (critical) behavior without apparently requiring the fine-tuning of parameters [4–6].

Sandpiles played a central role in the development of this field [1–3]. They are metaphors of real systems (as earthquakes, snow avalanches, stick-slip phenomena, etc.) in which some type of stress or energy is accumulated at some slow time scale and relaxed in a much faster way. In sandpiles, grains are slowly added until eventually they relax if a local instability threshold is overcome; then, they are transmitted to neighboring sites, which, in their turn, may become unstable and relax, generating avalanches of activity. Considering open boundaries to allow for energy balance, a steady state with power-law distributed avalanches is eventually reached.

In order to rationalize sandpiles in particular and SOC in general and to understand their critical properties, it was proposed [7–9] to look at them as systems with many absorbing states [10,11]. The underlying idea is that, in the absence of external driving, sandpile models get eventually trapped into stable configurations from which they cannot escape—i.e., absorbing states [10]. The concept of *fixed-energy sandpiles* (FESs) was introduced to make this connection more explicit. FESs share the microscopic rules with their standard (slowly driven and dissipative) counterparts, but with no driving or dissipation. In this way, the total amount of sand or energy becomes a conserved quantity acting as a control parameter. Calling E_c the average density of energy of a standard sandpile in its stationary (self-organized) critical state, it has been shown that the corresponding FES exhibits a transition from an active phase to an absorbing one at, precisely, E_c , while it is absorbing (active) below (above) E_c .

It can be argued that *slow driving and dissipation acting together at infinitely separated time scales constitute a mechanism able to pin a generic system with absorbing states and a conservation law to its critical point* [4,6–9,12].

Using the relation with transitions into absorbing states [13], a Langevin equation describing FES stochastic sandpiles was proposed [8,9] in the spirit of Hohenberg and Halperin [14]. The Langevin equation is similar to the well-known directed-percolation (DP) Langevin equation describing generic systems with absorbing states [10,15], but it is coupled linearly to a conserved nondiffusive energy field, representing the conservation of energy grains in the sandpile dynamics [8,9]:

$$\begin{aligned}\partial_t \rho(x,t) &= a\rho - b\rho^2 + \omega\rho E(x,t) + \nabla^2 \rho + \sigma\sqrt{\rho}\eta(x,t), \\ \partial_t E(x,t) &= D\nabla^2 \rho,\end{aligned}\quad (1)$$

where D , a , b , and ω are constants, $\rho(x,t)$ and $E(x,t)$ are the activity and the energy field, respectively, and η is a zero-mean Gaussian white noise. The universality class described by this Langevin equation, which includes sandpiles as well as all other systems with many absorbing states and a non-diffusive conserved field, is usually called the Manna or C-DP class [8,9,16,17]. As an important side note, let us remark that the original Bak-Tang-Wiesenfeld model, being deterministic, has many other conservation laws (toppling invariants) and is therefore not described by the present stochastic theory [9,18].

Incidentally, even if it has been clearly established that DP and C-DP constitute two different universality classes [19–21], most of their universal features (critical exponents, moment ratios, scaling functions, etc.) are very similar, making it difficult to discriminate numerically between both classes in any spatial dimension.

Despite the fact that the critical behavior of stochastic sandpiles is accepted to be universal and described by Eqs. (1), there are a few sandpile models which, despite being

conservative, have been argued to exhibit a different type of critical behavior [22,23]—namely, DP—violating the conjecture of universality.

(i) The *Mohanty-Dhar* (MD) sandpile [22], in which grains have some probability to remain stable even if they are above the instability threshold. It was argued, based on numerical simulations and on exact results for an anisotropic version of it, that the (isotropic) MD sandpile should exhibit DP behavior.

(ii) The *Maslov-Zhang* (MZ) sandpile [23], in which not only the grains of unstable sites are redistributed among nearest neighbors. Instead, all the grains in the local neighborhood of unstable sites are randomly redistributed or “reshuffled.” Various versions of the model (called “charitable,” “neutral,” and “greedy”) were defined depending on the bias in the local redistribution rule (“charitable” if the central site receives less than each nearest neighbor, “neutral” if all the sites in the neighborhood are treated alike, and “greedy” otherwise). On the basis of Monte Carlo simulations this sandpile (in its neutral version) was argued to be DP-like.

Aimed at clarifying this puzzling situation, in a recent work we provided strong numerical and analytical evidence that, contrarily to what previously thought, the (isotropic) MD sandpile is actually in the C-DP class [24] (see [26] for a discrepant viewpoint). To further substantiate our claim, in [25] we proposed a strategy to easily discriminate between DP and C-DP consisting in introducing a wall (either absorbing or reflecting); systems in the DP and in the C-DP classes behave in (qualitative and quantitatively) very different ways in the presence of walls, providing an easy criterion to discriminate between both classes.

Our goal in the present paper is to scrutinize the last remaining piece in the puzzle—i.e., the MZ model—by employing extensive numerical simulations as well as field-theoretical considerations. For the numerics, we take advantage of the previously introduced discrimination criterion and, also, present another method to easily discriminate between DP and C-DP, based in the introduction of anisotropy in one spatial direction.

The analyses presented in what follows show in a clean-cut way that the MZ model, in any of the three versions above, is actually in the C-DP class. This leaves no dangling end in the sandpile universality picture and, as a by-product, confirms the general validity of the absorbing-state approach to rationalize SOC. Results for the C-DP class are also obtained.

II. MASLOV-ZHANG SANDPILE

The (neutral) MZ sandpile [23] is defined as follows.

(i) Driving: an input energy $\delta E \leq 1$ is added to the central (or to a randomly chosen) site i of a d -dimensional lattice, and the site is declared active.

(ii) Relaxation: energy is locally redistributed (“reshuffled”) between the active site and its nearest neighbors, according to

$$E_i = \frac{r_i}{\sum_{j=1}^{2d+1} r_j} \sum_{j=1}^{2d+1} E_j, \quad (2)$$

where r_i are uniformly distributed ($r_i \in [0, 1]$) random variables and the sums are performed over the site i and its $2d$

nearest neighbors. This rule needs to be slightly modified for the “charitable” and “greedy” versions of the model that we will not explore in detail here.

(iii) Activation: each of the sites involved in the reshuffling is declared active with a probability given by its own energy, triggering the generation of avalanches of activity.

(iv) Avalanches: new active sites are added to a list and relaxed in a sequential way.

(v) Dissipation: energy arriving at the open borders is removed from the system.

(vi) Avalanches proceed until all activity ceases, and then a new external input is added. Eventually a critical stationary state is reached.

Monte Carlo simulations by Maslov and Zhang revealed exponents compatible with those of directed percolation in two dimensions and above, while in one dimension some anomalies in the scaling were reported [23]. Later, simulations of the FES counterpart of the MZ sandpile led to very similar results [8].

Before entering a more careful numerical analysis of the MZ sandpile, we take a detour to construct explicitly a Langevin equation for such a model. This will allow us to better understand what the main relevant ingredients of the MZ model are and in what sense it differs from other C-DP models and from Eqs. (1).

III. LANGEVIN EQUATION FOR THE MASLOV-ZHANG SANDPILE

The main difference between the MZ cellular automaton and other sandpiles in the C-DP universality class is that, while in the C-DP class as the Manna model, the only energy redistributed by the dynamics (by topplings) is that accumulated in active sites, in the MZ dynamics of both the energy of active sites as well as that of its nearest neighbors is redistributed. This leads to a more severe local redistribution of energy, which we quantify in what follows in terms of a new set of Langevin equations, first in a phenomenological way and then by deriving it from the microscopic rules.

A. Phenomenological Langevin equation

Diffusion of energy occurs in the C-DP class by means of activity relaxation. At a mesoscopic level, this implies that the rate of change of the energy density $E(x, t)$ at a given position is proportional to minus the divergence of a current, $\partial_t E(x, t) = -\nabla \cdot \mathbf{j}(x, t)$, where $j(x, t)$ is given by the gradient of the activity field,

$$j(x, t) = -D \nabla \rho(x, t), \quad (3)$$

leading to $\partial_t E(x, t) = D \nabla^2 \rho(x, t)$ in Eqs. (1).

Instead, in the MZ model, changes of energy are controlled by the reshuffling rule; i.e., energy does not need to be at an active site (but in its neighborhood) to be redistributed. Hence, at a mesoscopic level the current is given by *gradients of the energy in the presence of nonvanishing activity*. More specifically, $\partial_t E(x, t) = -\nabla \cdot \mathbf{j}(x, t)$ where now

$$j(x, t) = -\tilde{D} \nabla E(x, t), \quad (4)$$

and the diffusion \tilde{D} is not a constant but a functional proportional to $\rho(x, t)$, $\tilde{D}(\rho(x, t)) = D\rho(x, t)$, capturing the requirement that local reshuffling of energy only occur in the presence of activity. This enforces the absorbing state condition that dynamics ceases if $\rho=0$. Finally,

$$\partial_t E(x, t) = D \nabla \cdot [\rho(x, t) \nabla E(x, t)]. \quad (5)$$

On the other hand, the equation for the activity is not expected to change in any relevant way, so one obtains

$$\partial_t \rho(x, t) = a\rho - b\rho^2 + \nabla^2 \rho + \omega \rho E(x, t) + \sigma \sqrt{\rho} \eta(x, t),$$

$$\partial_t E(x, t) = D \nabla \cdot [\rho(x, t) \nabla E(x, t)], \quad (6)$$

representing the MZ dynamics at a mesoscopic level. This is to be compared with Eqs. (1).

B. Microscopic derivation of the Langevin equation

To gain more confidence in the phenomenological set of equations (6) we present here an explicit microscopic derivation. We consider a parallel version of the MZ model in which all active sites are relaxed at every time step. To do so, we assume that each site uses a fraction $1/(2d+1)$ of its total energy for eventual redistributions with each of the sites in its local neighborhood. At the mean-field level, the energy evolves according to

$$E_{i,t+1} = \frac{1}{2d+1} E_{i,t} \sum_{j=1}^{2d+1} [1 - \Theta(\rho_{j,t})] + \frac{1}{2d+1} \sum_{j=1}^{2d+1} \Theta(\rho_{j,t}) \sum_{k=1}^{2d+1} \frac{E_{k,t}}{2d+1}, \quad (7)$$

where j (k) runs over the nearest neighbors of i (j) and Θ is the Heaviside step function [$\Theta(z)=0$ if $z \leq 0$]. For any inactive site in the local neighborhood (i.e., $\rho_j=0$), the central site does not redistribute the corresponding fraction of its energy [first term in Eq. (7)]. For any active site in the local neighborhood (i.e., $\rho_j > 0$) the central site receives on average a corresponding fraction of energy (second term). Equation (7) is a mean-field equation to which fluctuations should be added to have a more detailed description. Such fluctuations appear as a (conserved) noise for the resulting energy equation and can be easily argued to constitute a higher-order, irrelevant, correction.

Regularizing the Heaviside step function in Eq. (7) by means of a hyperbolic tangent and expanding it in power series up to first order in ρ [24], we obtain

$$E_{i,t+1} = \frac{1}{2d+1} E_{i,t} \sum_{j=1}^{2d+1} [1 - \rho_{j,t}] + \frac{1}{2d+1} \sum_{j=1}^{2d+1} \rho_{j,t} \sum_{k=1}^{2d+1} \frac{E_{k,t}}{2d+1}. \quad (8)$$

Introducing the d -dimensional discrete Laplacian $\nabla^2 E_i = \sum_{j=1}^{2d} (E_j - E_i)$ the first term can be rewritten as

$$\frac{E_i}{2d+1} \sum_{j=1}^{2d+1} [1 - \rho_j] = E_i - \frac{E_i}{2d+1} \sum_{j=1}^{2d+1} \rho_j = E_i - E_i \rho_i - \frac{E_i}{2d+1} \nabla^2 \rho_i. \quad (9)$$

Similarly, the second term can be expressed as

$$\frac{1}{(2d+1)^2} \sum_{j=1}^{2d+1} \rho_j \left(\sum_{k=1}^{2d+1} E_k \right) = E_i \rho_i + \frac{1}{(2d+1)} \nabla^2 (E_i \rho_i) + \frac{1}{(2d+1)} \rho_i \nabla^2 E_i + \frac{1}{(2d+1)^2} \nabla^2 (\rho_i \nabla^2 E_i). \quad (10)$$

Putting these two contributions together and reorganizing the discrete derivatives, one obtains

$$\begin{aligned} E_{i,t+1} &= E_{i,t} + \frac{2}{2d+1} \rho_{i,t} \nabla^2 E_{i,t} + \frac{2}{2d+1} \nabla \rho_{i,t} \cdot \nabla E_{i,t} + \frac{1}{(2d+1)^2} \nabla^2 (\rho_{i,t} \nabla^2 E_{i,t}) \\ &= E_{i,t} + \frac{2}{2d+1} \nabla \cdot (\rho_{i,t} \nabla E_{i,t}) + \frac{1}{(2d+1)^2} \nabla^2 (\rho_{i,t} \nabla^2 E_{i,t}), \end{aligned} \quad (11)$$

which in the continuous-time limit becomes

$$\partial_t E_i = \frac{2}{2d+1} \nabla \cdot (\rho_{i,t} \nabla E_{i,t}) + \frac{1}{(2d+1)^2} \nabla^2 (\rho_{i,t} \nabla^2 E_{i,t}). \quad (12)$$

The second term in this last equation can be argued to be naively irrelevant in the renormalization-group sense (as it

includes higher-order derivatives) and hence dropped out. Identifying D with $2/(2d+1)$ we recover the phenomenological equation (5) as the leading contribution.

As said above, this is to be compared with the standard Langevin equations (1) for the C-DP class, in which the flux of energy is given by the gradient of the activity itself. The following questions pop up naturally: Does Eq. (6) lead to a critical behavior different from that of Eq. (1)? Could this be

TABLE I. Critical exponents for DP, C-DP, and the MZ sandpile in one dimension. DP and C-DP values taken from [28,20,25].

$d=1$	η	δ	τ	τ_t	z_{spr}	θ
DP	0.313(1)	0.159(1)	1.108(1)	1.159(1)	1.265(1)	0.159(1)
C-DP	0.350(5)	0.170(5)	1.11(2)	1.17(2)	1.39(1)	0.125(1)
MZ	0.32(5)	0.20(5)	1.13(5)	1.20(5)	1.40(5)	0.13(1)

the reason why the MZ sandpile was claimed to differ from the C-DP class? Is this new form of conserved energy dynamics irrelevant at the DP fixed point, supporting the MZ sandpile to be DP like?

To properly answer these questions one should resort to a full renormalization-group calculation. Given that even for the C-DP class this has proven to be a, still not satisfactorily accomplished, difficult task [27], we will leave aside such a strategy here. Instead, in Sec. IV we will give an answer to these question by means of computational studies of the microscopic MZ model as well as of its equivalent Langevin equations (6).

C. Naive power counting

A power-counting analysis is not helpful in elucidating the relevancy of the energy-diffusion term, Eq. (5). Actually, as there is a linear dependence on the energy field on both the right- and left-hand sides of Eqs. (1), there is no way to extract the energy field naive dimension or, hence, to make any statement about the relevancy of the coupling term $w\rho(x,t)E(x,t)$ at the DP fixed point.

On the other hand, it is easy to cast Eqs. (6) into a generating functional following standard procedures, to set the basis of a perturbative expansion. However, one soon realizes that technical difficulties similar to those encountered for the analysis of Eqs. (1) (including the presence of generically singular propagators [8,27]) show up, hindering the perturbative calculation. It is our believe that some type of nonperturbative technique, or nonconventional perturbative expansion, is required to elucidate the renormalization-group fixed point of this type of problems.

The analytical understanding, at a field theory level of C-DP as well as the MZ-Langevin equation, remains an open challenging task.

IV. MONTE CARLO SIMULATIONS OF THE MASLOV-ZHANG SANDPILE

We have performed extensive Monte Carlo simulations of the MZ sandpile (and variations of it) and scrutinized its asymptotic (long-time and large-system-size) properties. We report on two different types of numerical experiments.

(i) “SOC” or avalanche experiments. By iterating slow addition of grains in the sandpile with open boundaries, the system self-organizes to a state with average energy E_c . Then, the avalanche size distribution $P(s)$ and the avalanche time distribution $P_t(t)$ can be estimated and their corresponding exponents τ and τ_t [28] measured.

(ii) Absorbing state experiments. At the stationary state, we perform spreading experiments from a localized seed, and we measure (a) the mean quadratic distance to the initial seed, $R^2 \sim t^{z_{spr}}$, in active runs, (b) the average number of active sites as a function of time, $N(t) \sim t^\eta$, and (c) the surviving probability up to time t , $P_s \sim t^{-\delta}$. We also study the decay at criticality of a homogeneous initial activity ($\rho(t) \sim t^{-\theta}$) in the fixed-energy case [10,28].

Scaling laws relating avalanche to spreading exponents were described systematically in [28]; two of them are $\tau = (1 + \eta + 2\delta)/(1 + \eta + \delta)$ and $\tau_t = 1 + \delta$. We measure the exponents independently and use scaling laws as a check for consistency.

We simulate the MZ automaton in one-dimensional lattices up to size $L=2^{15}$. The stationary critical energy density is $E_c=0.4928(2)$, and contrarily to what reported in [23], we do observe clean scaling at criticality, even if it emerges only after significantly long transients (results not shown), justifying why smaller-scale simulations can lead to erroneous conclusions.

The resulting critical exponents are gathered together in Table I. They are closer in all cases to the C-DP values than to DP ones. The exponents measured explicitly in [23] are α [$\int dt N(t)/P_s(t) \sim t^\alpha$] and the fractal dimension D_f , which take also very similar values in both cases [$\alpha=1.47(1)$, $D_f=2.32(1)$ for DP, $\alpha=1.52(1)$, $D_f=2.36(1)$ for C-DP]. Still, given the similarity between the numerical values in both classes, it is not safe to extract a definitive conclusion from these results. Higher numerical precision would be required to produce fully convincing evidence. As said above, the numerical values of DP and C-DP exponents are closer and closer as the dimensionality is increased (they coincide above $d_c=4$). Therefore, performing numerical simulations, as the ones above, to discriminate between both classes by using larger and larger times and system sizes in $d \geq 2$, is not a clever idea. Instead, it is advisable to use and devise more effective numerical strategies to discriminate between DP and C-DP in a simple, efficient, and numerically inexpensive way. For this we use (a) the method devised in [25] consisting in analyzing how the system responds to the presence of a wall or (b) a strategy, which exploits the fact that systems in these two classes react in remarkably different ways to the introduction of anisotropy in space.

Both of these strategies allow to obtain clean-cut results, as shown in the forthcoming two subsections.

A. Boundary-driven experiments

The influence of walls in systems in the DP class has been profusely analyzed in the literature [29]. In particular, it is

TABLE II. One-dimensional critical exponents for DP and C-DP without walls [20,28] and in the presence of absorbing and reflecting walls [25]. Values in rows 1 (DP_{ref}) and 2 (DP_{abs}) coincide within error bars. Note also that values in row 3 ($C-DP_{ref}$) coincide with those for C-DP (Table I). Results for the MZ sandpile in the presence of reflecting and absorbing walls are reported in the last two rows.

$d=1$	η	δ	τ	τ_t	z_{spr}
DP_{ref}	0.046(2)	0.425(2)	1.25(3)	1.43(3)	1.257(2)
DP_{abs}	0.045(2)	0.426(2)	1.28(3)	1.426(2)	1.276(2)
$C-DP_{ref}$	0.35(1)	0.16(1)	1.11(2)	1.15(2)	1.41(1)
$C-DP_{abs}$	-0.33(2)	0.85(2)	1.56(2)	1.81(2)	1.43(2)
MZ_{ref}	0.37(5)	0.15(5)	1.09(5)	1.14(5)	1.30(5)
MZ_{abs}	-0.36(5)	0.84(5)	1.57(5)	1.84(5)	1.25(5)

well known that, if spreading (and SOC) experiments are performed nearby a wall, the surviving probability is significantly affected, and avalanche and spreading exponents change in a nontrivial way with respect to their bulk counterparts [29]. It is also known that in the DP class, both reflecting and absorbing walls lead to a common type of universal “surface critical behavior,” which we call surface-directed percolation (SDP), characterized (in one dimension) by the exponents shown in Table II.

In contrast, the effect of walls in C-DP systems has been studied only recently [25]. Contrarily to the DP case, absorbing and reflecting walls induce different types of surface critical behavior. As illustrated in Table II, all spreading and avalanche exponents take distinct values for an absorbing and for a reflecting wall. Furthermore, the numerical differences between the exponents for either type of wall with respect to their corresponding SDP counterparts are very large, allowing for easy numerical discrimination [25]. Finally, in the C-DP class, the exponents in the presence of a reflecting wall coincide with their bulk counterparts [25]. These features imply that, by introducing a wall in a given system with absorbing states, it becomes straightforward to distinguish if it is in the DP or in the C-DP class, with moderate computational cost.

Following this strategy, we simulated the one-dimensional MZ sandpile, as defined above, in the presence of both reflecting and absorbing walls. In both cases a wall is introduced at the origin (position $i=0$) and the sandpile is studied in the positive half lattice. In the *reflecting* case, the energy that should go after reshuffling, to the leftmost site, at $i=0$ (whose energy is fixed to zero), is instead added to its closest nearest neighbor to the right, $i=1$. On the other hand, the *absorbing* condition is imposed by fixing the energy of the leftmost site to zero after every iteration of the microscopic sandpile rules—i.e., by removing from the system at every iteration all the energy received by the leftmost site.

Figures 1 and 2 show the results of simulations performed in lattices of system size $L=2^{15}$, averaging over up to 8×10^7 runs. The corresponding exponents are summarized in Table II. All of them coincide within numerical accuracy with the expected values for the C-DP class in the presence of reflecting or absorbing walls, respectively, and differ sig-

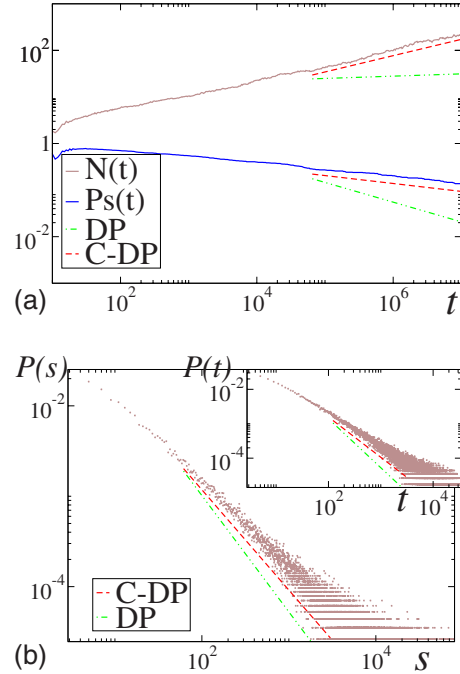


FIG. 1. (Color online) Avalanche exponents for the one-dimensional MZ sandpile in the presence of a *reflecting* wall. (a) Spreading experiments; $N(t)$ is the total number of active sites and $P_s(t)$ is the surviving probability; system size $L=2^{15}$ and number of runs, 500. (b) Avalanche size (main plot) and time (inset) distributions; system size $L=2^{12}$ and number of runs, 10^5 . Green (red) lines mark DP (C-DP) scaling (see numerical values in Table II).

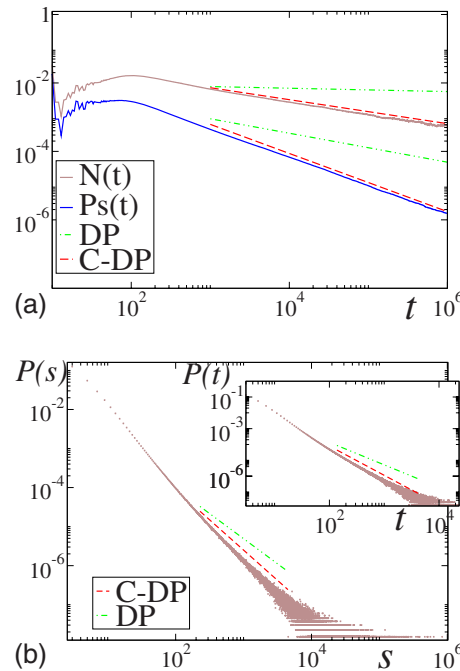


FIG. 2. (Color online) As in Fig. 1 but for an *absorbing* wall (numerical values summarized in Table II). System size $L=2^{15}$ and 8×10^7 runs.

TABLE III. One-dimensional critical exponents for DP [20,28], C-DP with a preferred direction (analytical results from [35]), and the anisotropic MZ model.

$d=1$	η	δ	τ	τ_t	z_{spr}
DP	0.33(2)	0.14(2)	1.09(3)	1.14(3)	2.00(2)
A-C-DP	0	1/2	4/3	3/2	2
A-MZ	-0.02(3)	0.51(3)	1.35(5)	1.48(5)	1.98(3)

nificantly from those of SDP. For example, in the presence of a reflecting (absorbing) wall, the measured value of η is 0.37 (-0.37), in good agreement with the C-DP expectation $\eta = 0.35$ (-0.33) and in blatant disagreement with the corresponding DP value $\eta \approx 0.046$ (≈ 0.045), which is one order of magnitude smaller (and of opposite sign in the case of an absorbing wall). Similar large differences are measured for all the exponents (see Table II). Note also that, as is the case in the C-DP class [25], the exponents in the presence of a reflecting wall coincide within error bars with their bulk counterparts. In conclusion, studying the influence of walls we conclude that the one-dimensional MZ sandpile exhibits C-DP scaling.

B. Anisotropic experiments

It is well known that systems in the DP class are invariant under Galilean transformations: if particles have a tendency to move anisotropically in one preferred spatial direction, that does not alter the critical properties [10]. The presence of any degree of anisotropy in DP-like systems is an irrelevant trait, or in other words, *anisotropic DP (A-DP) is just DP*.

The role of anisotropy in sandpiles has also been profusely studied after the pioneering exact solution by Dhar and Ramaswamy [30] of the totally anisotropic or “directed” counterpart of the Bak-Tang-Wiesenfeld sandpile. Anisotropic stochastic sandpiles have also been studied using general principles [31] and through interfacial representations [32]. The conclusion is that all anisotropic sandpiles, as long as they are stochastic [33], belong to the same universality class, which we call *anisotropic C-DP (A-C-DP)* [34]. The critical exponents of models in this class were first measured numerically [34] and then exactly calculated in any dimension [35] (see Tables III and IV).

The strategy to be used is straightforward: take the MZ sandpile model and switch on anisotropy; if the isotropic model was in the DP class, anisotropy should be an irrelevant

TABLE IV. Two-dimensional critical exponents for anisotropic models: A-DP (i.e., DP), A-C-DP (analytical results), and anisotropic-MZ.

$d=2$	η	δ	τ	τ_t	z_{spr}
DP	0.230(1)	0.451(1)	1.268(1)	1.451(1)	1.13(2)
A-C-DP	0	3/4	10/7	7/4	2
A-MZ	-0.07(5)	0.75(5)	1.49(5)	1.70(5)	1.93(5)

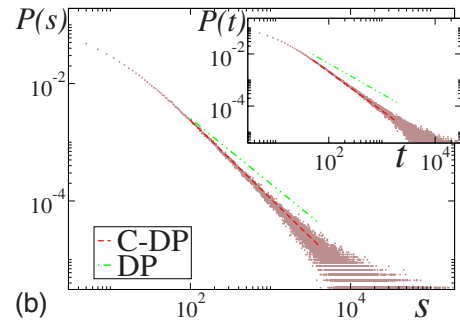
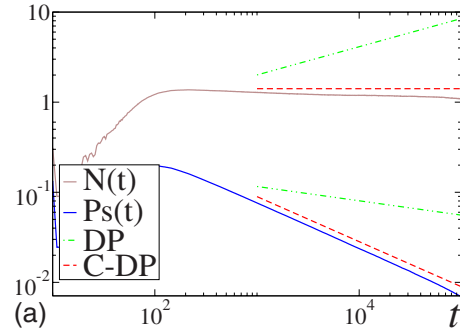


FIG. 3. (Color online) Avalanche exponents for the *anisotropic* MZ sandpile model in one dimension, averaged over 8×10^6 runs (system size $L=2^{18}$). (a) Spreading experiments (see Table III). (b) Avalanche size (main plot) and time (inset) distributions. Green lines mark DP scaling, while red ones correspond to C-DP scaling.

ingredient and the anisotropic counterpart should also be DP-like. If, instead, the isotropic model is in the C-DP class, then anisotropy is a relevant ingredient and critical exponents change from C-DP to A-C-DP values.

The simplest way to define an anisotropic MZ (A-MZ) model is by fixing one of the r_j in Eq. (2)—say, the one to the right—to its maximum possible value $r_j=1$ and letting the others r_j to take randomly distributed values in $[0,1]$. This generates an overall energy flow towards the preferred direction (to the right, in this case). Anisotropy can be introduced in other ways, including full anisotropy or directness, but this does not affect our conclusions in any significant way.

Figure 3 and Table III show our main results for the one-dimensional MZ model with anisotropy. Both avalanche and spreading exponents are very different from their isotropic counterparts. They also differ notoriously from DP values, but coincide within error bars with the expected values for the A-C-DP class. The same conclusion holds in two dimensions (see Table IV). In this way, as the anisotropic MZ model belongs to the A-C-DP class, the original, isotropic, MZ sandpile model can be safely concluded to be in the C-DP universality class, confirming the result above.

V. NUMERICAL INTEGRATION OF THE MZ LANGEVIN EQUATION

In this section we verify that the Langevin equations (6) are a sound description of the MZ model and that, despite of its different form, it behaves asymptotically as Eq. (1). For

TABLE V. Critical exponents for Eqs. (6), with a reflective wall, Eqs. (6) with an absorbing wall, and Eqs. (6) with an anisotropic term.

$d=1$	η	δ	τ	τ_l	z_{spr}	θ
Eqs. (6)	0.28(5)	0.21(5)	1.14(5)	1.21(5)	1.25(5)	0.14(2)
Eqs. (6) _{ref}	0.36(5)	0.18(5)	1.12(5)	1.18(5)	1.29(5)	0.11(5)
Eqs. (6) _{abs}	-0.39(5)	0.85(5)	1.58(5)	1.85(5)	1.22(5)	0.15(5)
Eqs. (6) _{anis}	-0.01(2)	0.50(2)	1.39(5)	1.64(5)	1.98(2)	0.51(2)

that we perform a numerical analysis (again, both SOC and absorbing state experiments) using Eqs. (6). A direct integration of Eqs. (6) in one dimension, using the recently introduced integration scheme for Langevin equations with square-root noise [21], produces the exponents reported in the first row of Table V (plots not shown). All of them are compatible with those of the microscopic MZ model and the C-DP class (see Table I). Changing the boundary conditions during the integration we implement the reflecting or the absorbing wall. For the former, we impose $\rho(-x, t) = \rho(x, t)$ and $E(-x, t) = E(x, t)$, while for the absorbing walls $\rho(x \leq 0, t) = 0$ and $E(x \leq 0, t) = 0$. The measured exponents, performing avalanche and spreading experiments nearby a reflecting (absorbing) wall at 0 (results not shown), are summarized in the second and third rows of Table V. Again, the exponents coincide within error bars with their corresponding C-DP counterparts and exclude DP scaling (see Table II).

Finally, we have studied an anisotropic version of the equations by introducing a term proportional to $\nabla\rho(x, t)$ into both the activity and the energy equations (6), obtaining again excellent agreement with the one-dimensional C-DP values (Table III).

In summary, we have integrated numerically Eqs. (6) and implemented the necessary modifications (i.e., include boundaries or anisotropy) to perform the tests described in the previous section. The obtained results are in excellent agreement with those for the microscopic model, confirming that (a) the Langevin equation derived in Sec. II is representative of MZ model and that (b) the MZ model is in the C-DP class.

VI. CONCLUSION AND DISCUSSION

We have shed some light on the picture of universality in stochastic sandpiles by confirming that, indeed, they all share the same universal critical behavior. As hypothesized some years ago, their critical features are captured by the set of Langevin equations (1), C-DP, describing in a minimal way the phase transition into a multiply degenerated absorbing state in the presence of a nondiffusive conserved field.

We have shown that the Maslov-Zhang sandpile, believed before to exhibit a different type of scaling (directed percolation like), is actually in the C-DP class, in agreement with the universality hypothesis. To reach this conclusion we have

performed large-scale simulations and introduced numerical strategies to easily discriminate between DP and C-DP. In particular, we have benefited from the fact that the, otherwise very similar, DP and C-DP classes behave in radically different ways both in the presence of walls and when anisotropy is switched on.

We have also derived, in two different ways, an alternative set of Langevin equations (6) describing the Maslov-Zhang sandpile. This set of equations is characterized by a different form of local energy diffusion [the corresponding current is proportional to energy gradients and not to activity gradients as is the case in Eqs. (1)]. By direct integration of the stochastic differential set of equations (6), we have shown that it describes the same universality class as Eqs. (1)—i.e., C-DP—despite of the formal differences in their respective equations for the conserved field, hence leading to a coherent global picture for the universality of sandpiles. This result actually enlarges the C-DP universality class, allowing one to embrace also different types of energy relaxation or redistribution dynamics, which includes also “charitable” versions of the MZ model. Instead, the “greedy” version, characterized by “antidiffusion” (i.e., energy accumulates on active sites), is expected to be highly anomalous.

Our analyses have several general implications for the C-DP universality class.

(i) Reflecting walls are not a relevant perturbation in this class: avalanche and spreading exponents measured in the vicinity of a reflecting wall coincide with their corresponding bulk counterparts. The underlying reason for this remains to be well understood.

(ii) Absorbing walls are relevant ingredients and affect the corresponding surface critical behavior. In particular, avalanches and spreading experiments performed nearby the wall are characterized by exponents that differ from their bulk counterparts.

(iii) Anisotropy in space is also a relevant ingredient. The corresponding critical behavior is described by the set of Langevin equations (1) [or, equivalently, Eqs. (6)] with an extra term $\nabla\rho(x, t)$ in both equations. Contrarily to the isotropic case, the critical exponents in the anisotropic class are known exactly in any dimension. The results coincide with those of anisotropic interfaces in random media, confirming once again the equivalence between the absorbing state and the interface pictures for SOC sandpiles [13].

It would be highly desirable to have a working renormalization-group calculation allowing one to put all the results discussed here on solid analytical grounds.

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- [1] P. Bak, C. Tang, and K. Wiesenfeld, *Phys. Rev. Lett.* **59**, 381 (1987); *Phys. Rev. A* **38**, 364 (1988).
- [2] D. Dhar, *Phys. Rev. Lett.* **64**, 1613 (1990); S. N. Majumdar and D. Dhar, *Physica A* **185**, 129 (1992).
- [3] S. S. Manna, *J. Phys. A* **24**, L363 (1991); P. Grassberger and S. S. Manna, *J. Phys. (Paris)* **51**, 1077 (1990); Y.-C. Zhang, *Phys. Rev. Lett.* **63**, 470 (1989); L. Pietronero, P. Tartaglia, and Y.-C. Zhang, *Physica A* **173**, 129 (1991); K. Christensen, A. Corral, V. Frette, J. Feder, and T. Jossang, *Phys. Rev. Lett.* **77**, 107 (1996).
- [4] G. Grinstein, in *Scale Invariance, Interfaces and Nonequilibrium Dynamics*, Vol. 344 of NATO Advanced Study Institute, Series B: Physics, edited by A. McKane *et al.* (Plenum, New York, 1995).
- [5] H. J. Jensen, *Self Organized Criticality* (Cambridge University Press, Cambridge, England, 1998); D. Dhar, *Physica A* **369**, 29 (1999); M. Alava, in *Advances in Condensed Matter and Statistical Physics*, edited by E. Korutcheva and R. Cuerno (Nova Science, New York, 2004), and references therein; D. L. Turcotte, *Rep. Prog. Phys.* **62**, 1377 (1999).
- [6] A. Vespignani and S. Zapperi, *Phys. Rev. Lett.* **78**, 4793 (1997); *Phys. Rev. E* **57**, 6345 (1998).
- [7] R. Dickman, A. Vespignani, and S. Zapperi, *Phys. Rev. E* **57**, 5095 (1998).
- [8] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, *Phys. Rev. Lett.* **81**, 5676 (1998); *Phys. Rev. E* **62**, 4564 (2000); R. Dickman, M. Alava, M. A. Muñoz, J. Peltola, A. Vespignani, and S. Zapperi, *ibid.* **64**, 056104 (2001). An early study of fixed-energy sandpiles is C. Tang and P. Bak, *Phys. Rev. Lett.* **60**, 2347 (1988).
- [9] R. Dickman, M. A. Muñoz, A. Vespignani, and S. Zapperi, *Braz. J. Phys.* **30**, 27 (2000); M. A. Muñoz, R. Dickman, R. Pastor-Satorras, A. Vespignani, and S. Zapperi, in *Modeling Complex Systems*, edited by J. Marro and P. L. Garrido, AIP Conf. Proc. No. 574 (AIP, Melville, NY, 2001), p. 102.
- [10] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions and Critical Phenomena* (Cambridge University Press, Cambridge, England, 1998); H. Hinrichsen, *Adv. Phys.* **49**, 1 (2000).
- [11] I. Jensen and R. Dickman, *Phys. Rev. E* **48**, 1710 (1993); M. A. Muñoz, G. Grinstein, R. Dickman, and R. Livi, *Phys. Rev. Lett.* **76**, 451 (1996); *Physica D* **103**, 485 (1997).
- [12] K. Christensen, N. R. Moloney, O. Peters, and G. Pruessner, *Phys. Rev. E* **70**, 067101 (2004).
- [13] An alternative is to map sandpiles onto models for the depinning of interfaces in random media: O. Narayan and A. A. Middleton, *Phys. Rev. B* **49**, 244 (1994); M. Paczuski, S. Maslov, and P. Bak, *Phys. Rev. E* **53**, 414 (1996); M. Paczuski and S. Boettcher, *Phys. Rev. Lett.* **77**, 111 (1996); M. Alava and K. B. Lauritsen, *Europhys. Lett.* **53**, 569 (2001); G. Pruessner, *Phys. Rev. E* **67**, 030301(R) (2003). These two approaches are actually equivalent: M. Alava and M. A. Muñoz, *ibid.* **65**, 026145 (2002); J. A. Bonachela, H. Chaté, I. Dornic, and M. A. Muñoz, *Phys. Rev. Lett.* **98**, 155702 (2007).
- [14] P. C. Hohenberg and B. J. Halperin, *Rev. Mod. Phys.* **49**, 435 (1977).
- [15] H. K. Janssen, *Z. Phys. B: Condens. Matter* **42**, 151 (1981); P. Grassberger, *ibid.* **47**, 365 (1982).
- [16] M. Rossi, R. Pastor-Satorras, and A. Vespignani, *Phys. Rev. Lett.* **85**, 1803 (2000). See also J. Kockelkoren and H. Chaté, e-print arXiv:cond-mat/0306039.
- [17] Equation (1) has also been derived rigorously by employing standard Fock-space formalism techniques for a reaction-diffusion model in this class [19].
- [18] L. P. Kadanoff, S. R. Nagel, L. Wu, and S. M. Zhou, *Phys. Rev. A* **39**, 6524 (1989); C. Tebaldi, M. De Menech, and A. L. Stella, *Phys. Rev. Lett.* **83**, 3952 (1999); O. Biham, E. Milshtein, and O. Malcai, *Phys. Rev. E* **63**, 061309 (2001).
- [19] R. Pastor-Satorras and A. Vespignani, *Phys. Rev. E* **62**, R5875 (2000); *Eur. Phys. J. B* **19**, 583 (2001).
- [20] See S. Lübeck, *Int. J. Mod. Phys. B* **18**, 3977 (2004), and references therein.
- [21] I. Dornic, H. Chaté, and M. A. Muñoz, *Phys. Rev. Lett.* **94**, 100601 (2005).
- [22] P. K. Mohanty and D. Dhar, *Phys. Rev. Lett.* **89**, 104303 (2002). See also B. Tadic and D. Dhar, *ibid.* **79**, 1519 (1997).
- [23] S. Maslov and Y.-C. Zhang, *Physica A* **223**, 1 (1996).
- [24] J. A. Bonachela, J. J. Ramasco, H. Chaté, I. Dornic, and M. A. Muñoz, *Phys. Rev. E* **74**, 050102(R) (2006).
- [25] J. A. Bonachela and M. A. Muñoz, *Physica A* **384**, 89 (2007).
- [26] P. K. Mohanty and D. Dhar, *Physica A* **384**, 34 (2007).
- [27] F. van Wijland, *Phys. Rev. Lett.* **89**, 190602 (2002); *Braz. J. Phys.* **33**, 551 (2003).
- [28] M. A. Muñoz, R. Dickman, A. Vespignani, and S. Zapperi, *Phys. Rev. E* **59**, 6175 (1999), and references therein.
- [29] See, P. Fröjdh, M. Howard, and K. B. Lauritsen, *J. Phys. A* **31**, 2311 (1998); *Int. J. Mod. Phys. B* **15**, 1761 (2001), and references therein.
- [30] D. Dhar and R. Ramaswamy, *Phys. Rev. Lett.* **63**, 1659 (1989).
- [31] G. Grinstein, D.-H. Lee, and S. Sachdev, *Phys. Rev. Lett.* **64**, 1927 (1990); T. Hwa and M. Kardar, *ibid.* **62**, 1813 (1989).
- [32] L.-H. Tang, M. Kardar, and D. Dhar, *Phys. Rev. Lett.* **74**, 920 (1995); S. Maslov and Y. C. Zhang, *ibid.* **75**, 1550 (1995); G. Pruessner and H. J. Jensen, *ibid.* **91**, 244303 (2003); G. Pruessner, *J. Phys. A* **37**, 7455 (2004).
- [33] Analogously to what happens with their isotropic counterparts, deterministic directed models behave in a different way than stochastic ones [34].
- [34] R. Pastor-Satorras and A. Vespignani, *Phys. Rev. E* **62**, 6195 (2000); *J. Phys. A* **33**, L33 (2000).
- [35] M. Paczuski and K. E. Bassler, *Phys. Rev. E* **62**, 5347 (2000); M. Kloster, S. Maslov, and C. Tang, *ibid.* **63**, 026111 (2001).